POSITIVITY PRINCIPLES AND DECAY OF SOLUTIONS IN SEMILINEAR ELLIPTIC PROBLEMS

VITALY MOROZ

ABSTRACT. This is an outline of the lectures delivered at the University of Rome Tor Vergata (May-June 2024) and at Zhejiang Normal University, Jinhua (October 2025). Please, email me if you spot any typos or mistakes.

CONTENTS

Consider the linear equation

$$(*) \qquad -\Delta u + Vu = f \quad \text{in } \Omega,$$

where Ω is a domain (open connected set) in $\mathbb{R}^N$ with $N \geq 2$ and V is a potential and $0 \leq f \in L^1_{loc}(\Omega)$ is a nonnegative right hand side. We assume that $V = V^+ - V^-$ and

$$V^+ \in L^\infty_{loc}(\Omega), \quad V^- \in L^1_{loc}(\Omega).$$

The natural quadratic form associated to the Schrödinger operator $-\Delta + V$ is given by

$$\mathcal{E}_V(\varphi) := \int_\Omega |\nabla\varphi|^2\,dx + \int_\Omega V\varphi^2\,dx \qquad (\varphi \in H^1_c(\Omega) \cap L^\infty_c(\Omega)),$$

where the subscript c denotes the *compact support*. We say that $\mathcal{E}_V$ is non-negative if

$$\mathcal{E}_V(\varphi) \geq 0, \qquad \forall\varphi \in H^1_c(\Omega) \cap L^\infty_c(\Omega).$$

Date: February 17, 2026.

We are going to study the relationship between the existence and some properties of positive supersolutions to $(*)$ and non-negativity of the quadratic form $\mathcal{E}_V$. Such a relationship is commonly referred to as Agmon–Allegretto–Piepenbrink's (AAP) Principle or "Ground State transformation". Our exposition is inspired and largely based on [4–7, 15] but is adapted with applications to semilinear equations in mind, as developed in [12–14].

1. The AAP positivity principle

A *weak* supersolution to $(*)$ is a function $u \in H^1_{loc}(\Omega) \cap L^1_{loc}(\Omega, V dx)$ such that

$$\int_\Omega \nabla u \cdot \nabla \varphi \, dx + \int_\Omega V u \varphi \, dx \geq \int_\Omega f \varphi \, dx, \qquad \forall\, 0 \leq \varphi \in H^1_c(\Omega) \cap L^\infty_c(\Omega). \tag{1.1}$$

The notions of a weak sub-solution and weak solution are defined similarly by replacing "$\geq$" with "$\leq$" and "$=$" respectively. Note that if $u \gneq 0$ is a weak supersolution to $(*)$ with $f \geq 0$ then $u > 0$ in Ω, in the sense that $u^{-1} \in L^\infty_{loc}(\Omega)$. This follows from the weak Harnack inequality which is ensured by the assumption $V^+ \in L^\infty_{loc}(\Omega)$.

In what follows we often omit the word *weak* and use simply solution, super and sub-solution. We say that u is a solution to $-\Delta + V$ if u is a solution for $(*)$ with $f = 0$. Similarly for sub and supersolutions.

When $V \geq 0$, the quadratic form $\mathcal{E}_V$ is non-negative, and any positive constant is a supersolution. Thus, the interesting case to consider is when V is negative or changes sign. The fundamental relation between the existence of positive supersolutions to $(*)$ and non-negativity of the quadratic form $\mathcal{E}_V$ is described in the following result, which was originally proved by W. Allegretto [2] and J. Piepenbrink [3] in 1974, and later became the foundation of S. Agmon's Criticality Theory [4].

Theorem 1.1 (AAP Positivity Principle). *Assume that $(*)$ admits a weak positive (super)solution $u_* > 0$. Then*

$$\mathcal{E}_V(\varphi)\,(\geq) = \int_\Omega \left|\nabla\left(\frac{\varphi}{u_*}\right)\right|^2 u_*^2 dx + \int_\Omega \varphi^2 \,\frac{f}{u_*}\, dx, \qquad \forall\, 0 \leq \varphi \in H^1_c(\Omega) \cap L^\infty_c(\Omega). \tag{1.2}$$

Proof. Let $0 \leq \varphi \in H^1_c(\Omega) \cap L^\infty_c(\Omega)$. Choose $\psi := \frac{\varphi^2}{u_*}$ as a test function for $(*)$ and note that

$$\nabla\left(\frac{\varphi^2}{u_*}\right) = \frac{2\varphi}{u_*}\nabla\varphi - \frac{\varphi^2}{u_*^2}\nabla u_* \in H^1_c(\Omega) \cap L^\infty_c(\Omega) \tag{1.3}$$

Testing $(*)$ against ψ we arrive at

$$\begin{aligned} \int_\Omega \nabla u_* \cdot \nabla\left(\frac{\varphi^2}{u_*}\right) dx + \int_\Omega V u_* \frac{\varphi^2}{u_*} dx &= \int_\Omega \left(\frac{2\varphi}{u_*}\nabla u_* \cdot \nabla\varphi \;-\; \frac{\varphi^2}{u_*^2}|\nabla u_*|^2\right) dx + \int_\Omega V \varphi^2\, dx \\ &\geq \int_\Omega \frac{f}{u_*}\varphi^2 \end{aligned} \tag{1.4}$$

Direct computation yields

$$\int_\Omega \left(|\nabla\varphi|^2 + V\varphi^2\right) dx - \int_\Omega \left|\nabla\left(\frac{\varphi}{u_*}\right)\right|^2 u_*^2 dx$$
$$= \int_\Omega \left(|\nabla\varphi|^2 + V\varphi^2\right) dx - \int_\Omega \left(\frac{|\nabla\varphi|^2}{u_*^2} - 2\varphi\,\nabla\varphi\,\frac{\nabla u_*}{u_*^3} + \varphi^2\frac{|\nabla u_*|^2}{u_*^4}\right) u_*^2 dx$$
$$= 2\int_\Omega \varphi\,\nabla\varphi\,\frac{\nabla u_*}{u_*}\,dx - \int_\Omega \varphi^2\frac{|\nabla u_*|^2}{u_*^2}\,dx + \int_\Omega V\varphi^2\,dx\,(\geq) = \int_\Omega \varphi^2\,\frac{f}{u_*}\,dx.$$

This proves (1.2) on $H^1_c(\Omega)\cap L^\infty_c(\Omega)$. □

The following straightforward corollary of Theorem 1.1 is crucial in the analysis of nonexistence of positive solutions to semilinear equations.

Corollary 1.2 (Nonexistence principle)**.** *Assume there exists $\varphi\in C^\infty_c(\Omega)$ such that $\mathcal{E}_V(\varphi)<0$. Then $-\Delta+V$ has no positive weak supersolutions in Ω.*

Remark 1.3. The computation in the proof of the AAP is often known as the *Picone's identity*:

$$|\nabla\varphi|^2 - \nabla\left(\frac{\varphi^2}{u}\right)\cdot\nabla u = \left|\nabla\left(\frac{\varphi}{u}\right)\right|^2 u^2 \geq 0,$$

for any $\varphi, v\in H^1_{loc}(\Omega)$ such that $\varphi\geq 0$, $u>0$ in Ω.

Remark 1.4. It is simple to see that the "u_*-ground state transformed" quadratic form

$$\mathcal{E}_{u_*}(\varphi) = \int_\Omega \left|\nabla\Big(\frac{\varphi}{u_*}\Big)\right|^2 u_*^2 dx$$

is generated by the nonnegative selfadjoint operator

$$-\Delta + 2\nabla\log(u_*)\nabla$$

in $L^2(\Omega, u_*^2\,dx)$.

Remark 1.5. (a) We do not assume any boundary conditions on the reference supersolution u_* in Lemma 1.1. We also do not require any boundary regularity of u_*. All we need is that $u_*\in H^1_{loc}(\Omega)$ is a positive weak supersolution of $-\Delta+V$. In many examples u_* is positive or even singular on parts of the boundary (Example 2.1), but u_* also may be zero on the boundary (Examples 2.4 and 2.6).

(b) We do not assume any boundary regularity of the potential V, which could be 'very' singular on $\partial\Omega$ (Examples 2.1, 2.4).

(c) Domain Ω need not be smooth (e.g. $\mathbb{R}^N\setminus\{0\}$ in Example 2.1).

2. Hardy type inequalities

In this section we will use the AAP positivity principle to prove several Hardy type inequalities. We start with the hardy inequality in $\mathbb{R}^N$.

Example 2.1 (Hardy inequality in $\mathbb{R}^N$)**.** Consider

$$-\Delta u - \frac{C_H}{|x|^2}u = 0 \quad \text{in } \mathbb{R}^N \setminus \{0\}, \tag{2.1}$$

where $N \geq 3$ and $C_H := (N-2)^2/4$. Clearly,

$$u_*(|x|) = |x|^{\frac{2-N}{2}}$$

is a positive weak solution of (2.1). Hence, by Lemma 1.1 we obtain Hardy's inequality

$$\int_{\mathbb{R}^N} |\nabla\varphi|^2 \geq C_H \int_{\mathbb{R}^N} \frac{\varphi^2}{|x|^2}, \qquad \forall \varphi \in C_c^\infty(\mathbb{R}^N \setminus \{0\}). \tag{2.2}$$

Remark 2.2. We had to remove the origin in (2.1) because $|x|^{\frac{2-N}{2}} \not\in H^1_{loc}(\mathbb{R}^N)$ – the singularity at the origin is too strong. To prove (2.1) in $\mathbb{R}^N$, first prove

$$\int_{\mathbb{R}^N} |\nabla\varphi|^2 \geq c \int_{\mathbb{R}^N} \frac{\varphi^2}{|x|^2}, \qquad \forall \varphi \in C_c^\infty(\mathbb{R}^N). \tag{2.3}$$

for any $c \in (0, C_H)$. To do this, use $u_* = |x|^{\alpha_+}$, where α_+ is the biggest root of the scalar equation $-\alpha(\alpha + N - 2) = c$. Since $\alpha_+ > -\frac{N-2}{2}$, we have $|x|^{\alpha_+} \in H^1_{loc}(\mathbb{R}^N)$ and the AAP principle is valid. Since (2.3) is valid for any $c < C_H$, it is also valid for $c = C_H$.

Exercise 2.3. Show that $C_H = \frac{(N-2)^2}{4}$ $(N \geq 3)$ is the *best constant* in the Hardy inequality (2.2) i.e. the inequality fails for any $c > C_H$.

Hint. Take a family of trial functions $\varphi_R = u_*\eta_R$ where $u_* = |x|^{\frac{2-N}{2}}$ and $\eta_R(r)$ is a non-negative C^1–cut-off function such that $\eta_R = 1$ for $r \in [0, R]$, $\eta_R = 0$ for $r > 2R$ and $|\eta'(r)| \leq 2/R$. Then use representation (1.2) to show that for any $\varepsilon > 0$,

$$\int_{\mathbb{R}^N} |\nabla\varphi_R|^2 - C_H \int_{\mathbb{R}^N} \frac{\varphi_R^2}{|x|^2} - \varepsilon \int_{\mathbb{R}^N} \frac{\varphi_R^2}{|x|^2} \to -\infty$$

as $R \to \infty$.

Example 2.4 (Hardy inequality in $\mathbb{R}^2$)**.** Consider

$$-\Delta u - \frac{1/4}{|x|^2\Big(\log\frac{|x|}{\rho}\Big)^2}u = 0 \quad \text{in } \mathbb{R}^2 \setminus B_\rho(0). \tag{2.4}$$

where $\rho > 0$. Clearly,

$$u_*(|x|) = \Big(\log\frac{|x|}{\rho}\Big)^{1/2}$$

is a positive weak solution of (2.4). Hence, by Lemma 1.1 we obtain a Hardy type inequality

$$\int_{\mathbb{R}^2} |\nabla\varphi|^2 \geq \frac{1}{4}\int_{\mathbb{R}^2} \frac{\varphi^2}{|x|^2\Big(\log\frac{|x|}{\rho}\Big)^2}, \qquad \forall \varphi \in C_c^\infty(\mathbb{R}^2 \setminus B_\rho(0)).$$

Exercise 2.5 (Improved Hardy inequality)**.** For $N \geq 2$ prove the following inequality

$$\int_{\mathbb{R}^N} |\nabla\varphi|^2 \geq C_H \int_{\mathbb{R}^N} \frac{\varphi^2}{|x|^2} + \frac{1}{4}\int_{\mathbb{R}^2} \frac{\varphi^2}{|x|^2(\log|x|)^2}, \qquad \forall\varphi \in C_c^\infty(\mathbb{R}^N \setminus B_1).$$

Hint: Take $u_*(|x|) = |x|^{-\frac{N-2}{2}}(\log|x|)^{1/2}$

Example 2.6 (Principal Dirichlet eigenvalue)**.** Let $\Omega \subset \mathbb{R}^N$ be a bounded domain, $\lambda_1 > 0$ be the principal Dirichlet eigenvalue of $-\Delta$ on Ω and $u_* := \phi_1 > 0$ be the corresponding principal eigenfunction. By Lemma 1.1 we obtain the inequality

$$\int_\Omega |\nabla\varphi|^2 \geq \lambda_1 \int_\Omega \varphi^2, \qquad \forall\varphi \in C_c^\infty(\Omega). \tag{2.5}$$

Example 2.7 (Torsional Hardy inequality)**.** Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and $\psi_\Omega > 0$ be a torsion function of Ω, that is the unique solution of

$$-\Delta\psi = 1 \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega.$$

Then we obtain "torsional Hardy inequality"

$$\int_\Omega |\nabla\varphi|^2 \geq \int_\Omega \frac{\varphi^2}{\psi_\Omega} \qquad \forall\varphi \in C_c^\infty(\Omega).$$

Now take $\varphi = \varphi_1$, the principal Dirichlet eigenvalue of Ω such that $\|\varphi_1\|_2 = 1$. Then

$$\lambda_1 = \int_\Omega |\nabla\varphi|^2 \geq \int_\Omega \frac{\varphi_1^2}{\psi_\Omega} \geq \frac{\|\varphi_1\|_2^2}{\|\psi_\Omega\|_\infty} = \frac{1}{\|\psi_\Omega\|_\infty}. \tag{2.6}$$

Hence we deduce a lower bound

$$\|\psi_\Omega\|_\infty \geq \frac{1}{\lambda_1}.$$

Exercise 2.8. For $R > 0$ and $B_R \subset \mathbb{R}^N$, prove the inequality

$$\int_{B_R} |\nabla\varphi|^2 \geq 2N \int_{B_R} \frac{\varphi^2}{R^2 - |x|^2} \qquad \forall\varphi \in C_c^\infty(B_R).$$

Exercise 2.9 (Barta's inequality)**.** Prove the following inequality

$$\inf_{x\in\Omega} \frac{(-\Delta - V)\phi}{\phi} \leq \inf_{0\neq u\in C_c^\infty(\Omega)} \frac{\mathcal{E}_V(u,u)}{\|u\|_{L^2}^2}. \tag{2.7}$$

Hint: Similar to (2.6).

3. The energy space

3.1. **The λ–property and the energy space.** Following [5], we say that $\mathcal{E}_V$ satisfies the *λ–property* if there exists a function $\lambda \in L^1_{\mathrm{loc}}(\Omega)$ with $\lambda > 0$ and $\lambda^{-1} \in L^\infty_{\mathrm{loc}}(\Omega)$ such that

$$\mathcal{E}_V(u) \geq \int_\Omega \lambda(x)\, u^2\, dx, \qquad \forall\, u \in C_c^\infty(\Omega). \tag{3.1}$$

If $\mathcal{E}_V$ satisfies the λ–property, then the form $\mathcal{E}_V$ is *coercive* and nondegenerate on $C_c^\infty(\Omega)$. Define

$$\langle u, v\rangle_V := \tfrac{1}{2}\big(\mathcal{E}_V(u+v) - \mathcal{E}_V(u) - \mathcal{E}_V(v)\big), \qquad \|u\|_V := \sqrt{\mathcal{E}_V(u)}.$$

Then $(C_c^\infty(\Omega), \langle\cdot,\cdot\rangle_V)$ is a pre–Hilbert space. Let $D^1_V(\Omega)$ denote its completion in the norm $\|\cdot\|_V$. The form $\mathcal{E}_V$ extends uniquely and continuously to $D^1_V(\Omega)$ by

$$\mathcal{E}_V(u) := \lim_{n\to\infty} \mathcal{E}_V(u_n),$$

for any sequence $(u_n) \subset C_c^\infty(\Omega)$ converging to u in $\|\cdot\|_V$; the limit is independent of the chosen approximating sequence. Thus $(D^1_V(\Omega), \langle\cdot,\cdot\rangle_V)$ is a Hilbert space, called the *energy space* of $\mathcal{E}_V$, and $\mathcal{E}_V$ is a closed, nonnegative quadratic form on it.

In view of (3.1),

$$\|u\|^2_{L^2(\Omega,\lambda\,dx)} = \int_\Omega |u(x)|^2\,\lambda(x)\,dx \leq \mathcal{E}_V(u),$$

and hence every $u \in D^1_V(\Omega)$ can be identified (up to sets of λ–measure zero) with an actual function $u \in L^2_{\mathrm{loc}}(\Omega)$, defined as the limit of an approximating Cauchy sequence $(\varphi_n) \subset C_c^\infty(\Omega)$ in the complete Hilbert space $L^2(\Omega, \lambda\,dx)$. In particular, the embedding $D^1_V(\Omega) \hookrightarrow L^2(\Omega,\lambda\,dx)$ is continuous. In this way, we proved the following.

Theorem 3.1. *The energy space $D^1_V(\Omega)$ is a Hilbert space with the inner product*

$$\langle u, v\rangle_V := \int_\Omega \nabla u\cdot\nabla v\,dx + \int_\Omega V uv\,dx$$

Moreover, $D^1_V(\Omega)$ is continuously embedded into $L^2(\Omega,\lambda\,dx)$.

Remark 3.2. Assume that the reference (ground state) function $u_* > 0$ in (1.2), used to define $\lambda = \frac{f}{u_*}$, is locally Lipschitz in Ω. Then $D^1_V(\Omega)$ is continuously embedded into $H^1_{\mathrm{loc}}(\Omega)$.

Indeed, let $\Omega' \Subset \Omega$. Following [10], we first observe that

$$\begin{aligned}\int_\Omega \Big|\nabla\Big(\frac{\varphi}{u_*}\Big)\Big|^2 u_*^2\,dx &= \int_\Omega \Big(|\nabla\varphi|^2 - 2\,\nabla\varphi\cdot\frac{\varphi}{u_*}\nabla u_* + \frac{\varphi^2}{u_*^2}|\nabla u_*|^2\Big)\,dx\\ &\geq \tfrac{1}{2}\int_\Omega \big(|\nabla\varphi|^2 - |\nabla\log u_*|^2\,\varphi^2\big)\,dx,\end{aligned}$$

where we used the Cauchy–Schwarz inequality

$$\nabla\varphi\cdot\frac{\varphi}{u_*}\nabla u_* \leq \frac{\varepsilon}{2}|\nabla\varphi|^2 + \frac{1}{2\varepsilon}\frac{\varphi^2}{u_*^2}|\nabla u_*|^2, \qquad \varepsilon = \tfrac{1}{2}.$$

Using (1.2), we then obtain

$$\begin{aligned}\frac{1}{2}\int_{\Omega'}|\nabla\varphi|^2\,dx &\le \int_{\Omega}\Big|\nabla\Big(\frac{\varphi}{u_*}\Big)\Big|^2 u_*^2\,dx + \frac{1}{2}\int_{\Omega'}|\nabla\log u_*|^2\,\varphi^2\,dx\\ &\le \mathcal{E}_V(\varphi) + c'_{\Omega'}\int_{\Omega'}\varphi^2\lambda(x)\,dx \le (1+c'_{\Omega'})\,\mathcal{E}_V(\varphi),\end{aligned}$$

where $c'_{\Omega'} > 0$ depends only on $\|\nabla\log u_*\|_{L^\infty(\Omega')}$. Hence $\|\varphi\|_{H^1(\Omega')} \le C_{\Omega'}\|\varphi\|_V$ for all $\varphi \in C_c^\infty(\Omega)$, and by density the embedding $D_V^1(\Omega) \hookrightarrow H^1_{\mathrm{loc}}(\Omega)$ follows.

Remark 3.3. If $\mathcal{E}_V$ satisfies the λ–property in Ω and $\Omega' \Subset \Omega$ is a bounded subdomain, then the same inequality

$$\mathcal{E}_V(u) \ge \int_{\Omega'}\lambda(x)\,u^2\,dx, \qquad u \in C_c^\infty(\Omega'),$$

holds in Ω', with $\lambda^{-1} \in L^\infty(\Omega')$. Consequently, the norms $\|u\|_V = \mathcal{E}_V(u)^{1/2}$ and the standard H_0^1–norm are equivalent on $C_c^\infty(\Omega')$, and thus the completion $D_V^1(\Omega')$ coincides with $H_0^1(\Omega')$.

Denote by $D_V^{-1}(\Omega)$ the dual space of all linear continuous functionals on $D_V^1(\Omega)$. The following lemma is a standard consequence of the Riesz Representation Theorem.

Lemma 3.4. *Assume that $\mathcal{E}_V$ satisfies the λ–property. Let $l \in D_V^{-1}(\Omega)$. Then there exists a unique $u_* \in D_V^1(\Omega)$ such that*

$$\langle u_*, \varphi\rangle = l(\varphi), \qquad \forall\,\varphi \in D_V^1(\Omega). \tag{3.2}$$

Since $D_V^1(\Omega) \subset L^2(\Omega,\lambda\,dx)$, by duality we conclude that $L^2(\Omega,\lambda^{-1}dx) \subset D_V^{-1}(\Omega)$, in the sense that if $f \in L^2(\Omega,\lambda^{-1}dx)$ then

$$l_f(\varphi) := \int_\Omega f\varphi\,dx \in D_V^{-1}(\Omega).$$

Thus Lemma 3.4 implies that for any $f \in L^2(\Omega,\lambda^{-1}dx)$ the equation

$$-\Delta u + Vu = f \quad \text{in } D_V^1(\Omega) \tag{3.3}$$

has a unique solution $u_f \in D_V^1(\Omega)$.

Exercise 3.5 (The space $H^1(\mathbb{R}^N)$)**.** Show that $-\Delta+1$ satisfies the λ–property in $\mathbb{R}^N$ and that the corresponding energy space $D_1^1(\mathbb{R}^N)$ coincides with the standard Sobolev space $H^1(\mathbb{R}^N)$.

Example 3.6 (The space $D_0^1(\mathbb{R}^N)$ for $N \ge 3$)**.** Consider $-\Delta$ on $\mathbb{R}^N$ with $N \ge 3$. Take

$$u_*(|x|) := (1+|x|^2)^{-\frac{N-2}{2}},$$

which is known as a Talenti function[1]. Then

$$-\Delta u^* = N(N-2)(1+|x|^2)^{-\frac{N+2}{2}}$$

[1]Up to a rescaling, it is a minimizer of the Sobolev inequality

and

$$\lambda(x) = \frac{-\Delta u_*(x)}{u_*(x)} = N(N-2)(1+|x|^2)^{-2}.$$

Hence the space $D^1_0(\mathbb{R}^N)$, a completion of $C^\infty_c(\mathbb{R}^N)$ with respect to the homogenous gradient norm $\|\nabla u\|_{L^2(\mathbb{R}^N)}$, is a well–defined Hilbert space and moreover,

$$D^1_0(\mathbb{R}^N) \subset L^2(\mathbb{R}^N, (1+|x|^2)^{-2}dx).$$

It is easy to see that $D^1_0(\mathbb{R}^N) \not\subset L^2(\mathbb{R}^N)$ and hence $D^1_0(\mathbb{R}^N) \neq H^1(\mathbb{R}^N)$. To see this when $N = 3, 4$, check that $\|\nabla u_*\|_{L^2(\mathbb{R}^N)} < \infty$ but $u_* \notin L^2(\mathbb{R}^N)$ when $N = 3, 4$. In fact, $D^1_0(\mathbb{R}^N) \subset L^{\frac{2N}{N-2}}(\mathbb{R}^N)$ – this follows from the Sobolev inequality.[2]

Example 3.7 (The space $D^1(\mathbb{R}^2)$ is not well-defined)**.** According to the Liouville's theorem, every positive super solution on $\mathbb{R}^2$ is constant. This means that the Laplacian $-\Delta$ does not satisfy λ–property on $\mathbb{R}^2$. Hence we can not apply Theorem 3.1 to construct the energy space $D^1_0(\mathbb{R}^2)$. In fact, such a space is not well defined as the space of functions, see [15] for details.

Example 3.8 (The space $D^1(\mathbb{R}^2 \setminus \bar{B}_\rho)$ in $\mathbb{R}^2$)**.** Consider $-\Delta$ on $\Omega = \mathbb{R}^2 \setminus \bar{B}_\rho$, for a $\rho > 0$. Take

$$u_*(|x|) = \left(\log \frac{|x|}{\rho}\right)^{1/2}.$$

As in the Example 2.4, we conclude that $-\Delta$ in Ω satisfies the λ–property with

$$\lambda(x) = \frac{1/4}{|x|^2\left(\log \frac{|x|}{\rho}\right)^2}.$$

Hence the space $D^1_0(\mathbb{R}^2 \setminus \bar{B}_\rho)$, the completion of $C^\infty_c(\mathbb{R}^2 \setminus \bar{B}_\rho)$ with respect to the homogenous gradient norm $\|\nabla u\|_{L^2(\mathbb{R}^2)}$ is a well–defined Hilbert space.

Exercise 3.9. Use the improved Hardy Inequality 2.5 to show that for $N \geq 3$ the critical Hardy operator

$$-\Delta - \frac{C_H}{|x|^2}, \qquad C_H := \left(\tfrac{N-2}{2}\right)^2,$$

satisfies the λ–property in $\mathbb{R}^N \setminus \bar{B}_1$ and define the energy space $D^1_{-\frac{C_H}{|x|^2}}(\mathbb{R}^N \setminus \bar{B}_1)$. Then use the arguments in Exercise 2.3 to show that

$$u_0 = |x|^{\frac{2-N}{2}} \in D^1_{-\frac{C_H}{|x|^2}}(\mathbb{R}^N \setminus \bar{B}_1).$$

Note that $u_0 \notin H^1_{loc}(\mathbb{R}^N)$!

[2]The space $D^1_0(\mathbb{R}^N)$ from this example is also often denoted as $D^1(\mathbb{R}^N)$ or $\dot{H}^1(\mathbb{R}^N)$.

3.2. **Maximum and comparison principles.** If $\mathcal{E}_V$ satisfies the λ–property then $-\Delta+V$ satisfies weak maximum and comparison principles. In order to prove that we need to know that $D^1_V(\Omega)$ is invariant under the standard truncations.

Lemma 3.10. *If $u \in D^1_V(\Omega)$ then $u^+, u^- \in D^1_V(\Omega)$ and*

$$\mathcal{E}_V(u^\pm) \le \mathcal{E}(u), \qquad \forall u \in D^1_V(\Omega). \tag{3.4}$$

If $u, v \in D^1_V(\Omega)$ then $u \vee v, u \wedge v \in D^1_V(\Omega)$.

Proof. See [14, Lemma A.1]. □

Remark 3.11. We do not claim that $u \in D^1_V(\Omega)$ implies $u \wedge 1 \in D^1_V(\Omega)$.

Exercise 3.12. Let $(\varphi_n) \subset C^\infty_c(\Omega)$ be an approximating sequence for $0 \le w \in D^1_V(\Omega)$. Show that $(\varphi_n^+) \subset D^1_V(\Omega)$ is also an approximating sequence for w, in the sense that $\mathcal{E}_V(\varphi_n^+) \to \mathcal{E}_V(w)$.

Hint. Show that (φ_n^-) is a Cauchy sequence and hence $\mathcal{E}_V(\varphi_n^-) \to 0$.

Lemma 3.13. *Assume that $\mathcal{E}_V$ satisfies the λ–property. Let $w \in H^1_{loc}(\Omega) \cap L^1(\Omega, V dx)$ be a super-solution to $(*)$ such that $w^- \in D^1_V(\Omega)$. Then $w \ge 0$ in Ω.*

Proof. Let $(\varphi_n) \subset C^\infty_c(\Omega)$ be an approximating sequence for $w^- \in D^1_V(\Omega)$. Set

$$w_n := \varphi_n^+ \wedge w^-.$$

Hence $0 \le w_n \in D^1_V(\Omega)$, by Lemma 3.10. Note that $w_n = w^- - (\varphi_n^+ - w^-)^-$. Therefore

$$\mathcal{E}_V(w^- - w_n) = \mathcal{E}_V((\varphi_n^+ - w^-)^-) \le \mathcal{E}_V(\varphi_n^+ - w^-) \to 0.$$

Thus (w_n) is a nonnegative approximating sequence for w^-. Since $0 \le w_n \le w^-$ we have $w^+ \wedge w_n = 0$, and then we obtain

$$0 \le \langle w, w_n \rangle_V = -\langle w^-, w_n \rangle_V \to -\mathcal{E}_V(w^-) \le 0.$$

We conclude that $w^- = 0$. □

The following comparison principle is a straightforward consequence of Lemma 3.13.

Corollary 3.14. *Assume that $\mathcal{E}_V$ satisfies the λ–property. Let $w \in H^1_{loc}(\Omega) \cap L^1(\Omega, V dx)$ be a super-solution to $(*)$ and $v \in H^1_{loc}(\Omega) \cap L^1(\Omega, V dx)$ be a sub-solution to $(*)$ such that $(w - v)^- \in D^1_V(\Omega)$. Then $w \ge v$ in Ω.*

Remark 3.15. In particular, Lemmas 3.4 and 3.13 imply that if $\mathcal{E}_V$ satisfies the λ–property then equation $(*)$ has a "rich" cone of positive super-solutions. Indeed, if $0 \lneq f \in L^2(\Omega, \lambda^{-1} dx)$ and $u_f \in D^1_V(\Omega)$ is the solution to (3.3) constructed in Lemma 3.4 then $u_f > 0$ in Ω by Lemma 3.13 and hence u_f is a weak positive supersolution for $-\Delta + V$. The situation is different if $\mathcal{E}_V$ is nonnegative but *does not* satisfy the λ–property.

Example 3.16 (Liouville's Theorem on the plane)**.** Consider $-\Delta$ on $\mathbb{R}^2$. Obviously, in this case the form $\mathcal{E}_0$ is nonnegative. Yet, according to the classical Liouville's theorem the only positive superharmonic for $-\Delta$ on $\mathbb{R}^2$ is constant.

Example 3.17 (Principal Dirichlet eigenvalue)**.** Let $\Omega \subset \mathbb{R}^N$ be a bounded domain, $\lambda_1 > 0$ be the principal Dirichlet eigenvalue of $-\Delta$ on Ω as in Example 2.6. By (2.5), the form $\mathcal{E}_{\lambda_1}$ is nonnegative. Yet, the the corresponding principal eigenfunction $\phi_1 > 0$ is (up to a constant) the only positive supersolution to $-\Delta - \lambda_1$ in Ω (see Corollary 1.2).

The above examples clarify the following classification:

- $-\Delta + V$ is *subcritical* if $\mathcal{E}_V$ satisfies the λ–property. In this case $-\Delta + V$ has a "rich" cone of positive supersolutions (see Remark 3.15).
- $-\Delta + V$ is *critical* if $\mathcal{E}_V$ is nonnegative but does not satisfy the λ–property. In this case $-\Delta + V$ has exactly one (up to a scalar) positive supersolution, which is actually a solution (see [4, Theorem 5.2]).
- $-\Delta+V$ is *supercritical* if $\mathcal{E}_V$ is not nonnegative. In this case $-\Delta+V$ has no positive supersolutions (see Corollary 1.2).

Further study of $-\Delta+V$ from the point of view of this classification is known as *criticality theory*, see [4,6,7]. In this lectures we are interested in one particular aspect only – we want to characterise the "size" of the cone of positive super-solutions of subcritical operators in terms of the admissible decay (or growth) of supersolutions "at infinity".

4. A Phragmén-Lindelöf principle

In this section we assume that Ω is an exterior domain in $\mathbb{R}^N$ such that $\mathbb{R}^N \setminus \bar{B}_1 \subset \Omega$ and $\partial\Omega \neq \emptyset$. We also always assume that $-\Delta + V$ satisfies the λ–property in Ω. Note that then $-\Delta + V$ also satisfies the λ–property in any subdomain $\Omega' \subset \Omega$, see Remark 3.3. We are going to study admissible decay at infinity, i.e. as $|x| \to \infty$, of all positive supersolutions to $-\Delta + V$ in Ω.

Definition 4.1. We say $u > 0$ is a *small* (sub) solution at infinity for $-\Delta + V$ if u is a (sub) solution for $-\Delta + V$ in B_R^c for some $R \geq 1$ and there exists a supersolution $w_* > 0$ for $-\Delta + V$ in Ω, such that

$$\lim_{|x|\to\infty} \frac{w_*}{u} = +\infty. \tag{4.1}$$

We say $U > 0$ is a *large* (sub) solution at infinity for $-\Delta + V$ in Ω if U is (sub) solution for $-\Delta+V$ in B_R^c for some $R \geq 1$ and U is *not* a small (sub) solution, i.e. for any supersolution $w > 0$ for $-\Delta + V$ in Ω,

$$\liminf_{|x|\to\infty} \frac{w}{U} < +\infty. \tag{4.2}$$

Exercise 4.2. Prove that if $u > 0$ is a small and $U > 0$ a large solution at infinity for $-\Delta + V$ in Ω then

$$\limsup_{|x|\to\infty} \frac{U}{u} = +\infty. \tag{4.3}$$

Intuitively, a small solution at infinity is "smaller" than one of the supersolutions. Then a *large* solution at infinity is *not* "smaller" than any of the supersolutions, i.e. in some sense it dominates at infinity all positive supersolutions. [3] An essential observation is that if a solution is smaller than *one* of the positive supersolutions in the sense of (4.1), then up to a constant it is dominated by *every* positive supersolution.

Lemma 4.3 (Small solution lemma)**.** *If $u > 0$ is a small solution at infinity for $-\Delta + V$ in Ω then for any supersolution $w > 0$ for $-\Delta + V$ in Ω there exists $c > 0$ such that*

$$u(x) \leq cw(x) \qquad (|x| \geq 1). \tag{4.4}$$

Proof. Choose $c > 0$ such that $cu \leq w$ as $|x| = 1$. For small $\varepsilon > 0$ consider the barrier functions

$$u_\varepsilon := cu - \varepsilon w_*.$$

In view of (4.1), there exists $\rho_\varepsilon > 1$ such that

$$u_\varepsilon \leq 0 \qquad (|x| \geq \rho_\varepsilon)$$

and moreover, $\rho_\varepsilon \to \infty$ as $\varepsilon \to 0$.

Set $\Omega_\varepsilon := \{x \in \mathbb{R}^N : 1 < |x| < \rho_\varepsilon\}$. Since $w > 0$, we conclude that

$$u_\varepsilon \leq w \quad \text{on } \partial\Omega_\varepsilon$$

and hence $(w - u_\varepsilon)^- \in H^1_0(\Omega_\varepsilon)$, see Remark 3.3. By the comparison principle of Corollary 3.14 we conclude that

$$u_\varepsilon \leq w \quad \text{in } \Omega_\varepsilon.$$

Then the assertion follows, since $\varepsilon > 0$ could be taken arbitrary small and taking into account that $\rho_\varepsilon \to \infty$ as $\varepsilon \to 0$. □

Proposition 4.4 (Phragmén-Lindelöf principle for supersolutions)**.** *Let $u > 0$ be a small and $U > 0$ a large solution at infinity for $-\Delta + V$ in Ω. Then for any supersolution $w > 0$ for $-\Delta + V$ in Ω:*

$$\liminf_{|x|\to\infty} \frac{w}{u} > 0, \qquad \liminf_{|x|\to\infty} \frac{w}{U} < \infty.$$

Proof. The first lim inf is simply the reformulation of (4.4), the second lim inf is the inversion of (4.2). □

Lemma 4.5 (Large solution lemma)**.** *Assume that $U > 0$ satisfies*

$$-\Delta U + VU = 0 \quad (|x| > 1) \quad \textit{and} \quad U = 0 \quad (|x| = 1). \tag{4.5}$$

Then U is a large solution for $-\Delta + V$.

Proof. Assume that U is not a large solution for $-\Delta + V$, i.e. U is a small solution. Then there exists a supersolution $w_* > 0$ for $-\Delta + V$ such that (4.1) holds.

For small $\varepsilon > 0$ consider the barrier functions

$$U_\varepsilon := U - \varepsilon w_*.$$

[3] Similarly, we could define small and large *sub*solutions but we will omit this for simplicity.

In view of (4.1), there exists $\rho_\varepsilon > 1$ such that

$$U_\varepsilon \le 0 \qquad (|x| \ge \rho_\varepsilon)$$

and moreover, $\rho_\varepsilon \to \infty$ as $\varepsilon \to 0$.

Set $\Omega_\varepsilon := \{x \in \mathbb{R}^N : 1 < |x| < \rho_\varepsilon\}$ and choose a small $\varepsilon > 0$ such that $U_\varepsilon^+ > \delta > 0$ on an open set $G \subset \Omega_\varepsilon$. Since $w_* > 0$ and $U_\varepsilon \le 0$ as $|x| = 1$ and $|x| = \rho_\varepsilon$, we conclude that for any $n \in \mathbb{N}$,

$$nU_\varepsilon \le w_* \quad \text{on } \partial\Omega_\varepsilon$$

and hence $(w_* - nU_\varepsilon)^- \in H_0^1(\Omega_\varepsilon)$, see Remark 3.3. By the comparison principle of Corollary 3.14 we conclude that

$$nU_\varepsilon \le w_* \quad \text{in } \Omega_\varepsilon.$$

Then $w_* \ge n\delta$ on G and since $n \in \mathbb{N}$ can be taken arbitrary large, $w_* = +\infty$ on G, which is a contradiction. □

Remark 4.6. If U satisfies (4.5) then $U \notin D_V^1(B_1^c)$. Indeed, assume $U \in D_V^1(B_1^c)$. Then $\mathcal{E}_V(U) = 0$ by (4.2). But $\mathcal{E}_V(u) > 0$ for any $0 \ne u \in D_V^1(B_1^c)$ since $\mathcal{E}_V$ satisfies the λ–property, a contradiction.

Remark 4.7. Assume that $U > 0$ is a (sub)solution in $|x| > 1$ and there exists a supersolution $w_* > 0$ such that

$$\lim_{|x|\to\infty} \frac{U}{w_*} = +\infty.$$

Then U is a large sub-solution at infinity.

Hint: Consider $U_* = U - cw_*$ for a sufficiently large $c > 0$.

Example 4.8 ($-\Delta$ in $\mathbb{R}^N$ with $N \ge 3$)**.** If $-\Delta w \ge 0$ in $\mathbb{R}^N \setminus \bar{B}_1$ then

$$\liminf_{|x|\to\infty} \frac{w(x)}{|x|^{-(N-2)}} > 0, \qquad \liminf_{|x|\to\infty} w(x) < \infty.$$

To see this, take $u(x) = |x|^{-(N-2)}$ as a small solution, $U(x) = 1 - |x|^{-(N-2)}$ as a large solution for $-\Delta$ in $\mathbb{R}^N \setminus \bar{B}_1$ and use Phragmén-Lindelöf principle for supersolutions. To check that $|x|^{-(N-2)}$ is a small solution, take $w_* = 1$ as a reference supersolution in (4.1).

Example 4.9 ($-\Delta$ in $\mathbb{R}^2$)**.** If $-\Delta w \ge 0$ in $\mathbb{R}^N \setminus \bar{B}_1$ then

$$\liminf_{|x|\to\infty} w(x) > 0, \qquad \liminf_{|x|\to\infty} \frac{w(x)}{\log|x|} < \infty.$$

To see this, take $u(x) = 1$ as a small solution, $U(x) = \log|x|$ as a large solution for $-\Delta$ in $\mathbb{R}^2 \setminus \bar{B}_1$ and use Phragmén-Lindelöf principle for supersolutions. To check that 1 is a small solution, take $w_* = \log|x|$ as a reference supersolution in (4.1).

Example 4.10 ($-\Delta + 1$ in $\mathbb{R}^3$)**.** If $-\Delta w + w \ge 0$ in $\mathbb{R}^3 \setminus \bar{B}_1$ then

$$\liminf_{|x|\to\infty} \frac{w(x)}{\frac{e^{-|x|}}{|x|}} > 0, \qquad \liminf_{|x|\to\infty} \frac{w(x)}{\frac{e^{|x|}}{|x|}} < \infty.$$

To see this, take

$$u(x) = \frac{e^{-|x|}}{|x|}, \qquad U(x) = \frac{e^{|x|-1} - e^{1-|x|}}{|x|},$$

as a small and large solution for $-\Delta + 1$ in $\mathbb{R}^3 \setminus \bar{B}_1$, respectively. Then use Phragmén-Lindelöf principle for supersolutions. To check that u is a small solution, take $w_* = 1$ as a reference supersolution in (4.1).

Exercise 4.11. Show that if

$$-\Delta w + \frac{c}{|x|^2} w \geq 0 \quad \text{in } \mathbb{R}^N \setminus \bar{B}_1,$$

and $c > -C_H := -\frac{(N-2)^2}{4}$, then

$$\liminf_{|x|\to\infty} \frac{w(x)}{|x|^{\alpha_-}} > 0, \qquad \liminf_{|x|\to\infty} \frac{w(x)}{|x|^{\alpha_+}} < \infty,$$

where $\alpha_- < \alpha_+$ are the two roots of the quadratic equation $\alpha(\alpha + N - 2) + c = 0$.

Note that if $c > 0$ then $\alpha_+ > 0$ and U is growing at infinity, while if $c < 0$ then $\alpha_+ < 0$ and U is decaying at infinity.

5. Half–space ansatz

Consider the following singularly perturbed linear problem model in the half-space

$$(A_{\mu,\varepsilon}) \qquad \begin{cases} -\Delta h = 0 & \text{in } \mathbb{R}^N_+, \\ \quad h = 0 & \text{on } \partial\mathbb{R}^N_+ \setminus \{0\}, \end{cases}$$

where $\mathbb{R}^N_+ = \{x \in \mathbb{R}^N \mid x_1 > 0\}$. We are going to construct explicit solutions of $(A_{\mu,\varepsilon})$.

Let

$$h(x) = x_1^\alpha r^\beta.$$

We compute

$$\begin{aligned} -\Delta h &= -\alpha(\alpha-1)x_1^{\alpha-2} r^\beta - \beta(\beta + N - 2 + 2\alpha) x_1^\alpha r^{\beta-2} \\ &= \left(\frac{-\alpha(\alpha-1)}{x_1^2} + \frac{-\beta(\beta+N-2+2\alpha)}{r^2} \right) h. \end{aligned}$$

If $\alpha_- \leq \alpha_+$ are as before the root of $\alpha(\alpha - 1) = \mu$ and

$$\beta_+ := -(N-2) - 2\alpha_+, \qquad \beta_- := -(N-2) - 2\alpha_-,$$

then the functions

$$h_+(x) = x_1^{\alpha_+} r^{\beta_+}, \qquad h_-(x) = x_1^{\alpha_-} r^{\beta_-},$$

are explicit solutions of the equation $(A_{\mu,0})$,

$$-\Delta h - \frac{\mu}{x_1^2} h = 0 \quad \text{in } \mathbb{R}^N_+.$$

Observe that these functions are distinct if and only if $\alpha_+ \neq \alpha_-$ or equivalently $\mu < \frac{1}{4}$.

We shall also need solutions of $(A_{\mu,\epsilon})$ with the same properties as $h_\pm$ on the boundary, especially when we are dealing with the case $\mu = 1/4$. Let $\beta_{\pm,-} \le \beta_{\pm,+}$ be the roots of

$$\beta(\beta + N - 2 + 2\alpha_\pm) = \varepsilon,$$

or equivalently, of

$$\beta\big(\beta + N - 1 \pm \sqrt{1-4\mu}\big) = \varepsilon. \tag{5.1}$$

Then any of the *four* functions

$$h_+(x) = x_1^{\alpha_+} r^{\beta_{+,+}}, \qquad h_-(x) = x_1^{\alpha_+} r^{\beta_{+,-}},$$
$$H_+(x) = x_1^{\alpha_-} r^{\beta_{-,+}}, \qquad H_-(x) = x_1^{\alpha_-} r^{\beta_{-,-}},$$

give a solution to $(A_{\mu,\varepsilon})$.

Remark 5.1. Although we are interested only in $\varepsilon \ge 0$, note that for a given $\mu \le 1/4$ solutions h_+ and h_- can be constructed for any

$$\varepsilon \ge -\Big(\frac{N-1}{2} + \sqrt{\frac{1}{4} - \mu}\Big)^2.$$

Via the AAP principle this leads to the family of Hardy type inequalities

$$\int_{\mathbb{R}^N_+} |\nabla\phi|^2 \ge \mu \int_{\mathbb{R}^N_+} \frac{\phi^2}{x_1^2} + \left(\frac{N-1}{2} + \sqrt{\frac{1}{4} - \mu}\right)^2 \int_{\mathbb{R}^N_+} \frac{\phi^2}{|x|^2}.$$

Interestingly, solutions H_+ and H_- are defined only for smaller range

$$\varepsilon \ge -\Big(\frac{N-1}{2} - \sqrt{\frac{1}{4} - \mu}\Big)^2,$$

and the latter constant vanishes if $\mu = -\frac{1}{4}N(N-2)$. Y. Pinchover pointed out that the constant is optimal. He refers to the arguments in the recent paper [21].

Lemma 5.2. (Minimal positive solution lemma) *Let $\mu < 1/4$. Then the problem*

$$L_\mu h_R = 0 \quad \text{in } \mathbb{R}^N_+ \cap B_R(0), \qquad h_R = x_1^{\alpha_-} \quad \text{on } \mathbb{R}^N_+ \cap S_R(0) \tag{5.2}$$

admits a positive solution such that for $x \in \mathbb{R}^N_+ \cap B_{R/2}(0)$ it holds

$$\limsup_{x_1 \to 0} \frac{h_R(x)}{x_1^{\alpha_+}} < \infty. \tag{5.3}$$

Proof. Let $\psi : [0, R] \to \mathbb{R}$ be a smooth function such that $\psi(R) = 1$, $0 \le \psi(r) \le 1$ and $\psi(r) = 0$ for $r \le R/2$. Set

$$f(x) := L_\mu\big(x_1^{\alpha_-}\psi\big) = x_1^{\alpha_-}\Big(-\psi'' - \frac{N-1+2\alpha_-}{r}\psi'\Big).$$

Clearly $f \in L^1(\mathbb{R}^N_+ \cap B_R(0), x_1^{\alpha_-}dx)$ and $f(x) = 0$ in $\mathbb{R}^N_+ \cap B_{R/2}(0)$. Then the problem

$$L_\mu \eta = -f \quad \text{in } \mathbb{R}^N_+ \cap B_R(0), \qquad \eta = 0 \quad \text{on } \partial(\mathbb{R}^N_+ \cap S_R(0))$$

admits a solution η such that for $x \in \mathbb{R}^N_+ \cap B_{R/2}(0)$ it holds

$$\limsup_{x_1 \to 0} \frac{\eta(x)}{x_1^{\alpha_+}} < \infty.$$

Then clearly

$$h_R := x_1^{\alpha_-}\psi + \eta$$

is the required solution of (5.2). □

We establish a version of the Phragmen–Lindelöf type comparison principle for sub-harmonics of L_μ, which shows that sub-harmonics either have a "strong" singularity at the origin or have a "minimal" decay at the origin. See [1, pp. 93-106] for a classical reference to the Phragmen–Lindelöf principle.

Lemma 5.3. (Phragmen–Lindelöf type estimate) *Let $\mu < 1/4$. Let h be an L_μ–sub-harmonic in $\mathbb{R}^N_+ \cap B_R(0)$, for some $R > 0$. Assume that $x \in \mathbb{R}^N_+ \cap B_R(0)$ and*

$$\lim_{x_1 \to 0} \frac{h(x)}{x_1^{\alpha_-} + x_1^{\alpha_+}|x|^{-(N-2+2\alpha_+)}} = 0. \tag{5.4}$$

Then for $x \in \mathbb{R}^N_+ \cap B_{R/2}(0)$ it holds

$$\limsup_{x_1 \to 0} \frac{h(x)}{x_1^{\alpha_+}} < \infty. \tag{5.5}$$

Proof. Without loss of generality we may assume that h is continuous on $\mathbb{R}^N_+ \cap \bar{B}_R(0)$. Let $h_R > 0$ be the minimal positive solution of (5.2), as constructed in Lemma 5.2. Note that

$$h_R(x) \le c x_1^{\alpha_+} \quad \text{in } \mathbb{R}^N_+ \cap B_{R/2}(0).$$

For $\tau > 0$, define a comparison function

$$h_\tau := h - h_R - \tau\big(x_1^{\alpha_-} + x_1^{\alpha_+}|x|^{-(N-2+2\alpha_+)}\big).$$

Clearly, h_τ is L_μ–sub-harmonic in $\mathbb{R}^N_+ \cap B_R(0)$.

For every $\tau > 0$, (5.4) combined with the construction of h_R implies that $h_\tau \le 0$ on a neighbourhood of $\partial(\mathbb{R}^N_+ \cap B_R(0))$. Hence we can apply the classical comparison principle for L_μ in a proper subdomain of $\mathbb{R}^N_+ \cap B_R(0)$ and deduce that $h_\tau \le 0$ everywhere in $\mathbb{R}^N_+ \cap B_R(0)$. By considering arbitrary small $\tau > 0$, we conclude that $h \le h_R$ in $\mathbb{R}^N_+ \cap B_R(0)$. □

Further reading. For the development of the criticality theory beyond Agmon's ideas in [4, 5] see e.g. [6–8]. Some Phragmén-Lindelöf type results in the context of linear elliptic equations can be found in [1]. Powerful applications to the analysis of Hardy type inequalities involving distance to the boundary are given in [9]. More recent developments of the criticality theory presented e.g. in [15, 21]. For an extension to p–Laplacian see [17]; to local and nonlocal Dirichlet forms see [19, 22]. This list is very far from being complete.

6. A nonlinear Liouville theorem

We are going to use the AAP and Phragmén–Lindelöf principles developed in the previous section to study positive supersolutions of the semilinear elliptic equation

$$-\Delta u = u^p \quad \text{in } \Omega, \tag{6.1}$$

where Ω is an exterior domain in $\mathbb{R}^N$ such that $\mathbb{R}^N \setminus \bar{B}_1 \subset \Omega$ and $\partial\Omega \neq \emptyset$, and $p \in \mathbb{R}$ is the nonlinear *exponent*, which could take both positive and negative values.

A *weak* (super) solution to (6.1) is a function $u \in H^1_{loc}(\Omega)$ such that $u^p \in L^1_{loc}(\Omega)$ and

$$\int_\Omega \nabla u \cdot \nabla\varphi \, dx \; (\geq) = \int_\Omega u^p \varphi \, dx, \qquad \forall\, 0 \leq \varphi \in H^1_c(\Omega) \cap L^\infty_c(\Omega). \tag{6.2}$$

Note that if $u \gneq 0$ is a weak supersolution to (6.1) then $-\Delta u \geq 0$ in Ω, and hence $u > 0$ in Ω. In what follows we often omit the *weak* and use simply (super) solution.

Theorem 6.1. *Let $N \geq 3$. Equation* (6.1) *admits a positive weak supersolution if and only if $p > p_S := \frac{N}{N-2}$.*

This theorem was proved by J. Serrin in the 70s for the radial functions, see the introduction to [11] for the references and a general overview of nonlinear Liouville's theorem. The exponent p_S is often known as the Serrin's critical exponent. The idea to use the AAP and Phragmén–Lindelöf principles in the context of nonlinear Liouville's theorems goes back to [12] and was developed in [13, 14, 16, 18, 20].

Proof. Our proof of Theorem 6.1 will be split into nonexistence and existence part. In the proof of nonexistence we will distinguish four separate cases: $1 < p < p_S$, $p = p_S$, $p = 1$, $p < 1$. The nonexistence in the superlinear case $1 < p < p_S$ relies on the *lower* bound in the Phragmén–Lindelöf principle while the nonexistence in the sublinear case uses the *upper* bound in the Phragmén–Lindelöf principle.

Before we start the proof, we present two technical lemmas. The first one is a particular case of the nonexistence counterpart of the AAP principle (Corollary 1.2).

Lemma 6.2. *Assume that $u > 0$ satisfies*

$$-\Delta u - c|x|^{-2+\varepsilon} u \geq 0 \quad (|x| > 1), \tag{6.3}$$

for some $\varepsilon > 0$ and $c > 0$. Then $u \equiv 0$.

Proof. Consider the quadratic form

$$\mathcal{E}(u) := \int_\Omega |\nabla\varphi|^2 \, dx - c \int_\Omega \frac{\varphi^2}{|x|^{2+\varepsilon}} \, dx \qquad (\varphi \in C^\infty_c(\Omega).$$

Take $\varphi \in C^\infty_c(\Omega)$ such that $\varphi \gneq 0$, $0 \leq \varphi \leq 1$ and $\varphi = 0$ for $|x| > 2$ and $|x| < 1$. Then for $R > 1$ the rescaling

$$\varphi_R(x) := \varphi\big(\tfrac{x}{R}\big) \in C^\infty_c(\Omega)$$

and by the change of variables we compute

$$\mathcal{E}(\varphi_R) = R^{N-2}\int_\Omega |\nabla\varphi|^2\,dx - cR^{N-2+\varepsilon}\int_\Omega \frac{\varphi^2}{|x|^{2-\varepsilon}}\,dx \to -\infty,$$

as $R\to\infty$. By the AAP principle (Corollary 1.2) we conclude that $u\equiv 0$. □

The second lemma is an an apriori lower bound on positive solutions of (6.1) in the sublinear case. Note that the bound (6.3) depends on the value of $p<1$.

Lemma 6.3. *Let $p<1$, $s\in\mathbb{R}$ and assume that $u>0$ satisfies*

$$-\Delta u \ge |x|^s u^p \quad \textit{in } \Omega.$$

Then

$$u \ge c|x|^{\frac{2+s}{1-p}} \quad (|x|>1). \tag{6.4}$$

Proof. The proof uses the AAP principle and weak Harnack's inequality, see [14, Lemma 6.1]. □

Now we are in a position to prove Theorem 6.1. We will proceed case by case.

Nonexistence in the superlinear subcritical case $1<p<p_S$. Assume that $u>0$ is a supersolution to (6.1). Then $-\Delta u\ge 0$ in Ω and hence, as in Example 4.8 we conclude that for some $c>0$,

$$u \ge c|x|^{-(N-2)} \quad (|x|>1) \tag{6.5}$$

Consider the *linearisation* of (6.1),

$$-\Delta u + V(x)u \ge 0 \quad \text{in } \Omega, \tag{6.6}$$

where $V(x):=u^{p-1}$. Since $p>1$, using (6.5) we can estimate

$$V(x) \ge c_1|x|^{-(N-2)(p-1)} \quad (|x|>1), \tag{6.7}$$

where $c_1=c^{p-1}>0$. Note that if $1<p<p_S$ then

$$-(N-2)(p-1) > -2. \tag{6.8}$$

Hence $u>0$ satisfies

$$-\Delta u + c_1|x|^{-2+\varepsilon}u \ge 0 \quad (|x|>1), \tag{6.9}$$

for some $\varepsilon>0$. By Lemma 6.2, we conclude that $u\equiv 0$. □

Nonexistence in the critical case $p=p_S$. In this case we have $\varepsilon=0$ in (6.9) and the previous argument fails.[4] Instead, we will iterate the previous to improve the lower bound (6.7).

Indeed, we may assume that $0<c_1<C_H$ and $u>0$ satisfies

$$-\Delta u - c_1|x|^{-2}u \ge 0 \quad (|x|>1). \tag{6.10}$$

[4]If $c_1>C_H$, the critical Hardy constant, we would conclude again by the AAP principle. However we do not control the size of $c_1>0$ and in general, c_1 could be small.

As in the Exercise 4.11 we conclude that for some $c_2 > 0$,

$$u \geq c_2|x|^{\alpha_-} \quad (|x| > 1), \tag{6.11}$$

where $\alpha_- > -(N-2)$ is the smallest root of $-\alpha(\alpha+N-2) = c_1$. Then $u > 0$ satisfies the linearisation equation (6.6) and we can estimate

$$V(x) \geq c_3|x|^{\alpha_-(p-1)} \quad (|x| > 1), \tag{6.12}$$

where $c_3 = c_2^{p-1}$. Since $p = p_S$ and $\alpha_- > -(N-2)$,

$$\alpha_-(p-1) > -2. \tag{6.13}$$

Hence $u > 0$ satisfies (6.9) with some $\varepsilon > 0$ and as before, by Lemma 6.2, we conclude that $u \equiv 0$. $\square$

Nonexistence in the linear case $p = 1$. In this case the equation (6.1) is linear. We simply note that there exists $\varphi \in C_c^\infty(\Omega)$ such that the corresponding quadratic form

$$\mathcal{E}_{-1}(\varphi) = \int_\Omega |\nabla\varphi|^2\,dx - \int_\Omega \varphi^2\,dx < 0.$$

(This can be seen using the same family of test functions φ_R as in the proof of Lemma 6.2.) By the AAP principle (Corollary 1.2) we conclude that $u \equiv 0$. $\square$

Nonexistence in the sublinear case $p < 1$. Assume that $u > 0$ is a supersolution to (6.1). Then $-\Delta u \geq 0$ in Ω and using the upper bound in Example 4.8 we conclude that

$$\liminf_{|x|\to\infty} u(x) < \infty. \tag{6.14}$$

But according to Lemma 6.3,

$$u \geq c|x|^{\frac{2}{1-p}} \quad (|x| > 1). \tag{6.15}$$

Since $p < 1$ these two bounds are incompatible with each other and we conclude that $u \equiv 0$. $\square$

Existence in the case $p > p_S$. A direct computation shows that for every $p > p_S$

$$u = c_p|x|^{-\frac{2}{p-1}}, \qquad c_p^{p-1} = \tfrac{2}{(p-1)^2}((N-2)p - N)$$

is a solution of (6.1).[5] $\square$

Exercise 6.4. Modify the previous arguments to show that if $N = 2$ then equation (6.1) has no positive weak supersolutions for any $p \in \mathbb{R}$.

Exercise 6.5. Let $N \geq 2$, $c > 0$ and $p \in \mathbb{R}$. Show that the equation

$$-\Delta u + \frac{c}{|x|^2}u = u^p \quad \text{in } |x| > 1$$

admits a positive weak supersolution if and only if $p \notin [1 - \frac{2}{\alpha_+}, 1 - \frac{2}{\alpha_-}]$, where α_- and α_+ are defined in Exercise 4.11.

[5] Note that $c_p < 0$ for $p < p_S$ and $c_p = 0$ if $p = p_S$.

Hint: Use small and large solutions constructed in Exercise 4.11. The nonexistence in the lower critical case $p = 1 - \frac{2}{\alpha_+} < 0$ is difficult, see [14, Lemma 6.6]. All other regimes could be studied similarly to the proof of Theorem 6.1.

Exercise 6.6. Let $N \geq 3$, $s > -2$ and $p \in \mathbb{R}$. Show that the equation

$$-\Delta u = |x|^s u^p \quad \text{in } |x| > 1$$

admits a positive weak supersolution if and only if $p > \frac{N+s}{N-2}$.

Hint: In the case $p < 1$ use (6.4).

Further reading. Similar methods based on the AAP and Phragmén–Lindelöf principles were used to prove Liouville's theorems for divergence type semilinear equations in conical domains [13], equations with Hardy type potentials in conical domains [14], quasilinear equations involving p-Laplacian [16], equations with Hardy type potentials involving distance to the boundary of a bounded domain [18] and nonlocal Choquard's equations [20].

Appendix A: Riesz potentials estimates

Here we present several estimates on the Riesz potentials of a reasonably fast decaying function that are fundamental in the applications.

Lemma 6.7. *Let $0 < \alpha < N$ and $0 \leq f \in L^1((1+|x|)^{\alpha-N}dx, \mathbb{R}^N)$. Let $0 \neq x \in \mathbb{R}^N$ and decompose $\mathbb{R}^N$ as the union of $A = \{y \notin B : |y| \leq |x|\}$, $B = \{y : |y-x| < |x|/2\}$, $C = \{y \notin B : |y| > |x|\}$. Then*

$$\int_{\mathbb{R}^N} \frac{f(y)}{|x-y|^{N-\alpha}}\, dy \;\simeq\; \frac{\int_A f(y)\, dy}{|x|^{N-\alpha}} + \int_B \frac{f(y)}{|x-y|^{N-\alpha}}\, dy + \int_C \frac{f(y)}{|y|^{N-\alpha}}\, dy. \tag{6.16}$$

Proof. We simply note that $|x|/2 \leq |x-y| \leq 2|x|$ for all $y \in A$, while $|y|/3 \leq |x-y| \leq 2|y|$ for all $y \in C$. To see the latter note that $|y-x| \leq |y| + |x| \leq 2|y|$ and that $2|y-x| \geq |x| \geq |y| - |y-x|$, so that $|y-x| \geq |y|/3$. □

Exercise 6.8. Show that $0 \leq f \in L^1((1+|x|)^{\alpha-N}dx, \mathbb{R}^N)$ is necessary and sufficient for the Riesz potential

$$\int_{\mathbb{R}^N} \frac{f(y)}{|x-y|^{N-\alpha}}\, dy$$

to be finite a.e. in $\mathbb{R}^N$.

Hint. Use Fubini to control the integral over B in (6.16).

Lemma 6.9. *Let $0 < \alpha < N$ and $0 \leq f \in L^1(\mathbb{R}^N)$. Assume that*

$$\lim_{|x|\to\infty} \frac{\int_{|y|\leq|x|} f(y)|y|\, dy}{|x|} = 0, \tag{6.17}$$

$$\int_{|y-x|\le|x|/2}\frac{f(y)}{|x-y|^{N-\alpha}}\,dy=o\big(|x|^{-(N-\alpha)}\big).\tag{6.18}$$

Then as $|x|\to\infty$,

$$\int_{\mathbb{R}^N}\frac{f(y)}{|x-y|^{N-\alpha}}\,dy=\frac{\|f\|_{L^1}}{|x|^{N-\alpha}}+o\big(|x|^{-(N-\alpha)}\big).\tag{6.19}$$

Note that $f\in L^1(\mathbb{R}^N)$ alone is not sufficient to obtain (6.19) even if f is radially symmetric, see [24].

Proof. Fix $0\neq x\in\mathbb{R}^N$ and decompose $\mathbb{R}^N$ as the union of $B=\{y:|y-x|<|x|/2\}$, $A=\{y\notin B:|y|\le|x|\}$, $C=\{y\notin B:|y|>|x|\}$.

We want to estimate the quantity

$$\left|\int_{A\cup C}f(y)\Big(\frac{1}{|x-y|^{N-\alpha}}-\frac{1}{|x|^{N-\alpha}}\Big)\,dy\right|\le\int_{A\cup C}f(y)\left|\frac{1}{|x-y|^{N-\alpha}}-\frac{1}{|x|^{N-\alpha}}\right|\,dy.\tag{6.20}$$

Since $|x|/2\le|x-y|\le 2|x|$ for all $y\in A$, by the Mean Value Theorem we have

$$\left|\frac{1}{|x-y|^{N-\alpha}}-\frac{1}{|x|^{N-\alpha}}\right|\le\frac{c_1|y|}{|x|^{N-\alpha+1}}\qquad(y\in A),\tag{6.21}$$

where $c_1=(N-\alpha)2^{N-\alpha+1}$. Thus

$$\left|\int_A f(y)\Big(\frac{1}{|x-y|^{N-\alpha}}-\frac{1}{|x|^{N-\alpha}}\Big)\,dy\right|\le\frac{c_1}{|x|^{N-\alpha+1}}\int_A f(y)|y|\,dy.\tag{6.22}$$

On the other hand, since $|x-y|>|x|/2$ for all $y\in C$ then

$$\left|\frac{1}{|x|^{N-\alpha}}-\frac{1}{|x-y|^{N-\alpha}}\right|\le\frac{1}{|x|^{N-\alpha}}\qquad(y\in C),\tag{6.23}$$

from which we compute that

$$\left|\int_C f(y)\Big(\frac{1}{|x-y|^{N-\alpha}}-\frac{1}{|x|^{N-\alpha}}\Big)\,dy\right|\le\frac{1}{|x|^{N-\alpha}}\int_C f(y)\,dy.\tag{6.24}$$

Then

$$\left|\int_{\mathbb{R}^N}\frac{f(y)}{|x-y|^{N-\alpha}}\,dy-\frac{\|f\|_{L^1}}{|x|^{N-\alpha}}\right|\le\tag{6.25}$$
$$\frac{c_1}{|x|^{N-\alpha+1}}\int_A f(y)|y|\,dy+\int_B\frac{f(y)}{|x-y|^{N-\alpha}}\,dy+\frac{1}{|x|^{N-\alpha}}\int_{B\cup C}f(y)\,dy.$$

The conclusion follows from (6.17), (6.18) and since $f\in L^1(\mathbb{R}^N)$. □

Corollary 6.10. *Let* $0<\alpha<N$ *and* $0\le f\in L^1(\mathbb{R}^N)$ *be a radially symmetric function that satisfies*

$$\lim_{|x|\to\infty}f(|x|)|x|^N=0.\tag{6.26}$$

If $\alpha\le 1$ *we additionally assume that* f *is monotone nonincreasing. Then* (6.19) *holds.*

Proof. Using (6.26) by l'Hospital rule we conclude that

$$\int_{|y|\le|x|} f(y)|y|\,dy = \int_0^{|x|} f(r)r^N dr = o(|x|) \qquad (|x|\to\infty), \tag{6.27}$$

so (6.17) holds.

For for $|x| \gg 1$ using radial estimates on the Riesz kernels in [25, Lemma 2.2] and (6.26) we obtain
for $\alpha > 1$:

$$\int_{|y-x|\le|x|/2} \frac{f(y)}{|x-y|^{N-\alpha}}\,dy \lesssim |x|^{\alpha-1}\int_{|x|/2}^{3|x|/2} f(r)dr = o(|x|^{-(N-\alpha)}); \tag{6.28}$$

for $\alpha = 1$, additionally using monotonicity of f:

$$\begin{aligned}\int_{|y-x|\le|x|/2} \frac{f(y)}{|x-y|^{N-\alpha}}\,dy &\lesssim \int_{|x|/2}^{3|x|/2} f(r)\log\frac{1}{1-r/|x|}dr \\ &\le f(|x|/2)\underbrace{\int_{|x|/2}^{3|x|/2}\log\frac{1}{1-r/|x|}dr}_{=3\log(3)|x|/2} = o(|x|^{-(N-1)});\end{aligned} \tag{6.29}$$

for $\alpha < 1$, additionally using monotonicity of f:

$$\begin{aligned}\int_{|y-x|\le|x|/2} \frac{f(y)}{|x-y|^{N-\alpha}}\,dy &\lesssim \int_{|x|/2}^{3|x|/2} \frac{f(r)}{|r-|x||^{1-\alpha}}dr \\ &\le f(|x|/2)\underbrace{\int_{|x|/2}^{3|x|/2}\frac{1}{|r-|x||^{1-\alpha}}dr}_{=c|x|^{\alpha}} = o(|x|^{-(N-\alpha)});\end{aligned} \tag{6.30}$$

so (6.18) holds. □

Appendix B: A Brezis–Browder type result

By modifying the proof of Lemma 3.13, we can establish the following approximation lemma, cf. [23, Theorem 3.4.1].

Lemma 6.11. *Assume that $\mathcal{E}_V$ satisfies the λ–property and $0 \le w \in D^1_V(\Omega)$. Then there exists a sequence $(w_n) \in D^1_V(\Omega) \cap L^\infty(\Omega)$ such that:*

(*a*) $\mathrm{supp}(w_n)$ *is compact in* Ω;

(*b*) $0 \le w_n \le w$;

(*c*) $\mathcal{E}_V(w - w_n) \to 0$ *as* $n \to \infty$.

Proof. Let $(\varphi_n) \subset C^\infty_c(\Omega)$ be an approximating sequence for $w \in D^1_V(\Omega)$. Set

$$w_n := \varphi_n^+ \wedge w \wedge n.$$

Then (w_n) has all the required properties and (c) follows as in the proof of Lemma 3.13. □

An important consequence of Lemma 6.11 is a Brezis–Browder type result, cf. [23, Theorem 3.4.1] and a historical discussion in [23, p.82].

Theorem 6.12. *Let* $T \in D_V^{-1}(\Omega) \cap L^1_{loc}(\Omega)$ *and* $w \in D_V^1(\Omega)$*. Assume*

$$T(x)w(x) \geq 0 \quad \text{a.e. in } \Omega. \tag{6.31}$$

Then $Tw \in L^1(\Omega)$ *and*

$$\langle T, w\rangle = \int_\Omega T(x)w(x)dx, \tag{6.32}$$

where $\langle\cdot,\cdot\rangle$ *denotes the duality between* $D_V^1(\Omega)$ *and* $D_V^{-1}(\Omega)$*.*

Proof. Let (w_n) be an approximating sequence for w, defined in Lemma 6.11. Since $T \in L^1_{loc}(\Omega)$ and $w_n \in L_c^\infty(\Omega)$, we know that

$$\langle T, w_n\rangle = \int_\Omega T(x)w_n(x)dx.$$

By Lemma 6.11 (c), $\langle T, w_n\rangle \to \langle T, w\rangle$ as $n \to \infty$. On the other hand, $T(x)w_n(x) \geq 0$ a.e. in Ω and then by Fatou's lemma $Tw \in L^1(\Omega)$. But we also know that $T(x)w_n(x) \leq T(x)w(x)$ a.e. in Ω. Then

$$\lim_{n\to\infty}\int_\Omega T(x)w_n(x)dx = \int_\Omega T(x)w(x)dx,$$

by dominated convergence. □

Exercise 6.13. Show that (6.31) can be replaced by one of the weaker assumptions:

$$T(x) \geq 0 \quad \text{a.e. in } \Omega, \tag{6.33}$$

or

$$T(x)w(x) \geq -|f(x)| \quad \text{a.e. in } \Omega, \tag{6.34}$$

for some $f \in L^1(\Omega)$.

Acknowledgement. VM is grateful to Roberto Ognibene, Paolo Cosentino and Daniel Raom for careful reading of the manuscript and their helpful comments which helped to improve the exposition.

References

[1] M. H. Protter and H. F. Weinberger, *Maximum principles in differential equations*, Springer-Verlag, New York, 1984. Corrected reprint of the 1967 original. ↑15

[2] W. Allegretto, *On the equivalence of two types of oscillation for elliptic operators*, Pacific J. Math. **55** (1974), 319–328. ↑2

[3] J. Piepenbrink, *Nonoscillatory elliptic equations*, J. Differential Equations **15** (1974), 541–550. ↑2

[4] S. Agmon, *On positivity and decay of solutions of second order elliptic equations on Riemannian manifolds*, Methods of functional analysis and theory of elliptic equations (Naples, 1982), Liguori, Naples, 1983, pp. 19–52. ↑2, 10, 15

[5] ———, *Bounds on exponential decay of eigenfunctions of Schrödinger operators*, Schrödinger operators (Como, 1984), Lecture Notes in Math., vol. 1159, Springer, Berlin, 1985, pp. 1–38. ↑2, 6, 15

[6] M. Murata, *Structure of positive solutions to* $(-\Delta + V)u = 0$ *in* $\mathbf{R}^n$, Duke Math. J. **53** (1986), no. 4, 869–943. ↑2, 10, 15

[7] Y. Pinchover, *On positive solutions of second-order elliptic equations, stability results, and classification*, Duke Math. J. **57** (1988), no. 3, 955–980. ↑2, 10, 15

[8] M. Murata, *On construction of Martin boundaries for second order elliptic equations*, Publ. Res. Inst. Math. Sci. **26** (1990), no. 4, 585–627. ↑15

[9] M. Marcus, V. J. Mizel, and Y. Pinchover, *On the best constant for Hardy's inequality in* $\mathbf{R}^n$, Trans. Amer. Math. Soc. **350** (1998), no. 8, 3237–3255. ↑15

[10] P. Takáč and K. Tintarev, *Generalized minimizer solutions for equations with the p-Laplacian and a potential term*, Proc. Roy. Soc. Edinburgh Sect. A **138** (2008), no. 1, 201–221. ↑6

[11] J. Serrin and H. Zou, *Cauchy-Liouville and universal boundedness theorems for quasilinear elliptic equations and inequalities*, Acta Math. **189** (2002), no. 1, 79–142. ↑16

[12] V. Kondratiev, V. Liskevich, and Z. Sobol, *Second-order semilinear elliptic inequalities in exterior domains*, J. Differential Equations **187** (2003), no. 2, 429–455. ↑2, 16

[13] V. Kondratiev, V. Liskevich, and V. Moroz, *Positive solutions to superlinear second-order divergence type elliptic equations in cone-like domains*, Ann. Inst. H. Poincaré Anal. Non Linéaire **22** (2005), no. 1, 25–43. ↑2, 16, 19

[14] V. Liskevich, S. Lyakhova, and V. Moroz, *Positive solutions to singular semilinear elliptic equations with critical potential on cone-like domains*, Adv. Differential Equations **11** (2006), no. 4, 361–398. ↑2, 9, 16, 17, 19

[15] Y. Pinchover and K. Tintarev, *A ground state alternative for singular Schrödinger operators*, J. Funct. Anal. **230** (2006), no. 1, 65–77. ↑2, 8, 15

[16] V. Liskevich, S. Lyakhova, and V. Moroz, *Positive solutions to nonlinear p-Laplace equations with Hardy potential in exterior domains*, J. Differential Equations **232** (2007), no. 1, 212–252. ↑16, 19

[17] Y. Pinchover and K. Tintarev, *Ground state alternative for p-Laplacian with potential term*, Calc. Var. Partial Differential Equations **28** (2007), no. 2, 179–201. ↑15

[18] C. Bandle, V. Moroz, and W. Reichel, *'Boundary blowup' type sub-solutions to semilinear elliptic equations with Hardy potential*, J. Lond. Math. Soc. (2) **77** (2008), no. 2, 503–523. ↑16, 19

[19] D. Lenz, P. Stollmann, and I. Veselić, *The Allegretto-Piepenbrink theorem for strongly local Dirichlet forms*, Doc. Math. **14** (2009), 167–189. ↑15

[20] V. Moroz and J. Van Schaftingen, *Nonexistence and optimal decay of supersolutions to Choquard equations in exterior domains*, J. Differential Equations **254** (2013), no. 8, 3089–3145. ↑16, 19

[21] B. Devyver, M. Fraas, and Y. Pinchover, *Optimal Hardy weight for second-order elliptic operator: an answer to a problem of Agmon*, J. Funct. Anal. **266** (2014), no. 7, 4422–4489. ↑14, 15

[22] R. L. Frank, D. Lenz, and D. Wingert, *Intrinsic metrics for non-local symmetric Dirichlet forms and applications to spectral theory*, J. Funct. Anal. **266** (2014), no. 8, 4765–4808. ↑15

[23] D. R. Adams and L. I. Hedberg, *Function spaces and potential theory*, Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 314, Springer-Verlag, Berlin, 1996. ↑21, 22

[24] D. Siegel and E. Talvila, *Pointwise growth estimates of the Riesz potential.* Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal. **5** (1999), 185-194. pages 20

[25] J. Duoandikoetxea, *Fractional integrals on radial functions with applications to weighted inequalities.* Ann. Mat. Pura Appl. (4) **192** (2013), no. 4, 553-568. pages 21

Department of Mathematics, Swansea University, Fabian Way, Swansea SA1 8EN, Wales, United Kingdom

Email address: v.moroz@swansea.ac.uk